Link Invariants from Classical Chern-Simons Theory

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Abstract

Taking as starting point a perturbative study of the classical equations of motion of the non-Abelian Chern-Simons theory with non dynamical sources, we obtain analytical expressions for link invariants. In order to present this expressions in a manifestly diffeomorphism-invariant form, we introduce a set of differential forms associated with submanifolds in R^3, that are metric independent, and that allow us to consider the link invariants as a kind of surface-dependent diffeomorphism invariants that present certain Abelian gauge symmetry.

I. INTRODUCTION

Since the discovery of the relation between Quantum Field Theories with Knot Theory, there has been an important progress both from the physical and the mathematical points of view. The starting point of this interplay was the recognition of the vacuum expectation value of the Wilson Loop in the Chern-Simons theory as a polynomial invariant of knots [1]. Soon, the perturbative study of the Wilson Loop average, using standard Feynmann rules, showed that every term in the perturbative series produces a knot invariant too [2].

As it was pointed out in a recent review [3], interesting issues coming from the physical side have their mathematical counterpart in Chern-Simons theory. For instance, gauge freedom is related with the fact that there exist different representations for knot invariants, corresponding to different gauge fixings. All of these developments have been performed within the Quantum Field Theory framework.

Recently, the author sketched a proposal to study link invariants from classical non-Abelian Chern-Simons theory [4], which is based on a previous work about the Abelian case [5]. That preliminary proposal was incomplete in at least to aspects. First, the action taken

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as starting point was not gauge invariant. Then, the results were forced to be gauge invariant by imposing a consistence condition *ad hoc*. Secondly, there was no *a posteriori* checking of the diffeomorphism invariance of the results obtained. The purpose of this article is to elaborate further on that program, and to remedy those aspects. To this end, we consider the classical equations of motion for the non-Abelian Chern-Simons field coupled to particles carrying non abelian charge (Wong particles) [6] , and argue that the on-shell action of this model should lead to analytical expressions for link invariants of the world lines of the particles. The action that we take is due to Balachandran *et al* [7], and, unlike the action that we employed in reference [5], it is gauge invariant. In view of the non linearity of the system, which prevents us to obtain exact solutions of the equations of motion, we develop a perturbative scheme to solve them. From it we explicitly obtain the first two contributions to the on-shell action. It is found that they correspond to the Second and Third Milnor Linking Coeficients, which are the first two in a family of link invariants of increasing complexity discovered by Milnor (the Second Coeficient coincides with the Gauss Linking Number) [8]. We also provide diffeomorphism covariant expressions for the link invariants obtained. To this end, we introduce a set of differential forms associated with volumes, surfaces, paths and points in R^3. These forms are easy to manipulate and also allow for a simple geometric interpretation of the link invariants obtained.

The paper is organized as follows. In section II we present the model. In section III we discuss the method for the obtention of link invariants from the solution of the classical equations of motion. Also, we consider the consistence conditions that the perturbative equations must obey in order to preserve gauge invariance. In section IV we introduce the differential forms mentioned above, and show that the link invariants obtained by our method can be cast into a manifestly diffeomorphism invariant form. Some final remarks are left for the last section.

II. CHERN-SIMONS-WONG THEORY

Our starting point will be the action

$$S = S_{CS} + S_{int}, \tag{1}$$

where

$$S_{CS} = -\Lambda^{-1} \int d^3x\, \epsilon^{\mu\nu\rho}\, Tr(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho) \tag{2}$$

is the $SU(N)$ Chern-Simons action and

$$S_{int} = \sum_{i=1}^{n} \int_{\gamma_i} d\tau\, Tr(K_i g_i^{-1}(\tau) D_\tau g_i(\tau)) \tag{3}$$

corresponds to the interaction, through the Chern-Simons field, between n particles of Wong, which are classical particles carrying non-abelian charge [6,7]. We take $Tr(T^a T^b) = -\frac{1}{2}\delta^{ab}$ and $T^a T^b = f^{abc} T^c$ for the $N^2 - 1$ generators T^a of the $SU(N)$ algebra. We shall use the notation $A_\mu = A^a_\mu T^a$, $A_i(\tau) = A_\mu(z_i(\tau)) \dot{z}^\mu_i(\tau)$. In eq. (3), the curve γ_i represents the world line of the $i-th$ particle. The $SU(N)$ matrix $g_i(\tau)$ is a dynamical variable, from which one can construct the chromo-electric charge $I_i(\tau)$ as

$$\begin{aligned} I_i(\tau) &\equiv g_i(\tau) K_i g_i^{-1}(\tau) \\ &= I_i^a(\tau) T^a, \end{aligned} \tag{4}$$

where $K_i \equiv K_i^a T^a$ is a constant element of the algebra, which, as we shall see, is related to the initial value of the chromo-electric charge $I_i(\tau)$. In eq. (3) also appears the covariant derivative of $g_i(\tau)$ along the world line of the $i-th$ particle

$$D_\tau g_i(\tau) = \dot{g}_i(\tau) + A_i(\tau) g_i(\tau). \tag{5}$$

The dynamical variables are the gauge potentials A_μ^a and the matrices $g_i(\tau)$ associated to the internal degrees of freedom of the Wong particles. One could also add to the action the usual contribution of the free particles

$$S_{particles} = -\int d\tau \sum_i \sqrt{\mid \dot{z}_i^\mu(\tau) \mid^2}, \tag{6}$$

and consider the trajectories $z_i^\mu(\tau)$ as dynamical objects too. However, this is not convenient for our purposes, since we want to take the curves γ_i as external objects whose linking properties are going to be studied. More precisely, we shall seek for link invariants related to closed curves in R^3. We shall take these curves just as the world lines of the Wong particles, which will follow externally prescribed trajectories. Furthermore, observe that in absence of the term given by eq.(6), the action that we take is topological, and this property is necessary in our program. It should be noticed that the relation of these particles with physically propagating ones is only formal, because, since we are interested in knotting properties of curves, we shall take the particles world lines as closed curves in Euclidean three space.

The Chern-Simons action is invariant under gauge transformations connected with the identity

$$A_\mu \rightarrow A_\mu^\Omega = \Omega^{-1} A_\mu \Omega + \Omega^{-1} \partial_\mu \Omega. \tag{7}$$

On the other hand, the action S_{int} is also gauge invariant provided that

$$K_i^\Omega = K_i, \tag{8}$$

$$g_i^\Omega = \Omega^{-1} g_i, \tag{9}$$

which in turn implies

$$(D_\tau g_i)^\Omega = \Omega^{-1} D_\tau g_i, \tag{10}$$

as corresponds to a covariant derivative. In view of the above equations, the non-Abelian charge transforms gauge-covariantly in the adjoint representation

$$I_i^\Omega = \Omega^{-1} I_i \Omega. \tag{11}$$

Varying the action (1) with respect to A_μ we obtain the field equation

$$\epsilon^{\mu\nu\rho} F_{\nu\rho} = \Lambda J^\mu, \tag{12}$$

where the current is given by

$$J^\mu(x) = \sum_{i=1}^{n} \int_{\gamma_i} d\tau \, \dot{z}_i^\mu(\tau) \, I_i(\tau) \, \delta^3(x - z_i(\tau)), \tag{13}$$

and the field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. On the other hand, to vary the action with respect to the $SU(N)$ elements $g(\tau)$, one must isolate the independent degrees of freedom. This can be acomplished by parametrizing the group elements as [7]

$$g((\xi(\tau)) = \exp(\xi^a(\tau) T^a), \tag{14}$$

and then varying the action with respect to the $N^2 - 1$ (for each particle) independent variables ξ^a . The resulting Euler-Lagrange equations

$$\frac{\partial L}{\partial \xi_i^a(\tau)} - \frac{d}{d\tau}\Big(\frac{\partial L}{\partial \dot{\xi}_i^a(\tau)}\Big) = 0, \tag{15}$$

where

$$L \equiv \sum_i Tr(K_i g_i^{-1}(\tau) D_\tau g_i(\tau)), \tag{16}$$

can be seen to be equivalent to gauge-covariant conservation of the non-Abelian charge of each particle along its world line [7]

$$D_\tau I_i = \dot{I}_i + [A_i, I_i] = 0. \tag{17}$$

The solution of this equation can be writen as

$$I_i(\tau) = U_i(\tau) \, I_i(0) \, U_i^{-1}(\tau), \tag{18}$$

where $U_i(\tau)$ is the time ordered exponential of the gauge potential along the world line γ_i

$$U_i(\tau) = Texp \, (- \int_0^\tau A_i(\tau') \, d\tau' \;). \tag{19}$$

It can be seen that equation (17), on the other hand, is necessary to fulfil the consistence condition that rises by taking the covariant derivative on both sides of equation (12). Summarizing, we have that the Chern-Simons-Wong model described by the action (1) is self-consistent, gauge-invariant and, if the world lines of the particles are externally given, also topological, in the sense that it is metric-independent.

III. LINK INVARIANTS

Let us suppose that we are able to solve the non linear equations of motion (12) under suitable boundary conditions, obtaining the gauge fields as functionals of the curves γ_i. Then, the on-shell action will be expressed in terms of these curves. But, since the action S is a metric-independent scalar function, the same will hold for the on-shell action S_{os}. Therefore, S_{os} has to be a metric-independent functional of curves, i.e., a link invariant. This should be compared with what occurs in the quantum evaluation of the vacuum expectation value of the Wilson Loop $W(C)$ [1,2]. There, the CS potential is integrated out, hence, $< W(C) >$ only depends on the curve C . Again, the metric independence of both the CS action and the Wilson Loop leads to conclude that the result must be a knot (or link) invariant.

Since we do not know how to solve the equation (12) exactly, we shall develop a perturbative solution. As it will be seen, this procedure leads to obtain the action on-shell as a power series in Λ

$$S_{os}\left([\gamma_i],\Lambda\right) = \sum_{p=0}^{\infty} \Lambda^p \; S^{(p)}[\gamma_i]. \tag{20}$$

where $S^{(p)}[\gamma_i]$, the $p-th$ coefficient in the expansion, carries the dependence on the curves γ_i. Now, if $S_{os}\left([\gamma_i],\Lambda\right)$ is a link invariant, so must be their derivatives with respect to Λ. Hence, the coefficients $S^{(p)}[\gamma_i]$ should be link invariants too. A useful consequence of this simple argument, which is also valid for the perturbative series of $< W(C) >$ in the quantum case [2], is that one does not need to get the whole power series in order to obtain link invariants. In this paper we shall study the first two invariants that this method provide.

From equations (4) and (18) we have

$$I_i(\tau) = U_i(\tau) g_i(0) K_i g_i^{-1}(0) U_i^{-1}(\tau). \tag{21}$$

Hence, we can take $g_i(\tau) = U_i(\tau) g_i(0)$, which implies

$$D_\tau g_i(\tau) = 0, \tag{22}$$

and then

$$S_{os}^{interaction} = 0. \tag{23}$$

Thus, it remains to consider S_{os}^{CS}. To proceed further, we find it convenient to rewrite the equation of motion (17) for the non-Abelian charge as

$$\frac{dI_i^a(\tau)}{d\tau} + \Lambda R_i^{ac}(\tau) I_i^c(\tau) = 0, \tag{24}$$

where we have defined

$$\begin{aligned} R_i^{ac}(\tau) &\equiv R_{i\mu}^{ac}(z_i(\tau))\dot{z}_i^{\mu}(\tau) \\ &\equiv f^{abc} B_\mu^b(\tau)\, \dot{z}_i^{\mu}(\tau), \end{aligned} \tag{25}$$

with

$$B_\mu = \Lambda^{-1} A_\mu. \tag{26}$$

Solving equation (24) we get

$$I_i^a(\tau) = \Big\{ T\,exp\Big(-\Lambda^{-1} \int_0^\tau d\tau' R_i(\tau') \Big) \Big\}^{ab} I_i^b(0), \tag{27}$$

which is another form of writing the result given by eqs. (18) and (19). Introducing (27) in the equation of motion (12), and expanding the time ordered exponential we arrive to the expression

$$\begin{aligned}
2\epsilon^{\mu\nu\rho}\,\partial_\nu B_\rho^a(x) = & -\Lambda\,\epsilon^{\mu\nu\rho}\,f^{abc} B_\nu^b(x) B_\rho^c(x) + \sum_{i=1}^n \oint_{\gamma_i} dz^\mu \delta^3(x-z)\, I_i^a(0) \\
& -\Lambda \sum_{i=1}^n \oint_{\gamma_i} dz^\mu \int_0^z dz_1^{\mu_1}\, R_{\mu_1}^{aa_1}(z_1)\, \delta^3(x-z)\, I_i^{a_1}(0) \\
& +\Lambda^2 \sum_{i=1}^n \oint_{\gamma_i} dz^\mu \int_0^z dz_1^{\mu_1} \int_0^{z_1} dz_2^{\mu_2}\, R_{\mu_1}^{aa_1}(z_1) R_{\mu_2}^{a_1a_2}(z_2) \delta^3(x-z)\, I_i^{a_2}(0) \\
& \vdots \\
& +(-\Lambda)^p \sum_{i=1}^n \oint_{\gamma_i} dz^\mu \int_0^z dz_1^{\mu_1} \ldots \int_0^{z_{p-1}} dz_p^{\mu_p}\, R_{\mu_1}^{aa_1}(z_1) \ldots R_{\mu_p}^{a_{p-1}a_p}(z_p) \delta^3(x-z) I_i^{a_p}(0) \\
& \vdots
\end{aligned} \tag{28}$$

In this equation, we substitute B_ρ^a by the power series

$$B_\rho^a = \sum_{p=0}^\infty \Lambda^p {B^{(p)}}^a_\rho, \tag{29}$$

which allows us to write the equation that the $p-th$ contribution ${B^{(p)}}^a_\rho$ to the potential obeys, in the form

$$\begin{aligned}
2\epsilon^{\mu\nu\rho}\,\partial_\nu {B^{(p)}}^a_\rho(x) = & -\epsilon^{\mu\nu\rho}\,f^{abc} \sum_{r,s=0}^{r+s=p-1} {B^{(r)}}^b_\nu(x) {B^{(s)}}^c_\rho(x) + \sum_{r=1}^p (-1)^r \sum_{i=1}^n \oint_{\gamma_i} dz^\mu \int_0^z dz_1^{\mu_1} \ldots \\
\ldots \int_0^{z_{r-1}} & dz_r^{\mu_r} \sum_{s_1,\ldots,s_r=o}^{s_1+\ldots+s_r=p-r} {R^{(s_1)}}^{aa_1}_{\mu_1}(z_1) {R^{(s_2)}}^{a_1a_2}_{\mu_2}(z_2) \ldots {R^{(s_r)}}^{a_{r-1}a_r}_{\mu_r}(z_r) \delta^3(x-z)\, I_i^{a_r}(0).
\end{aligned} \tag{30}$$

This equation holds for $p > 0$. For $p = 0$ one has, instead

$$2\epsilon^{\mu\nu\rho}\,\partial_\nu {B^{(0)}}^a_\rho(x) = \sum_{i=1}^n \oint_{\gamma_i} dz^\mu \delta^3(x-z)\, I_i^a(0). \tag{31}$$

Despite its complicate appearance, equation (30) has two nice features. Firstly, its right hand side involves $B^{(q)}$, with $q < p$, hence, one can look for a recursive solution. Secondly, the structure of eq.(30) is the same as that of the $0-th$ order equation (31): it is just like the Ampere's Law, whose solution is given by the Biot-Savart Law

$$B^{(p)}{}^{a}_{\alpha}(x) = -\frac{1}{4\pi}\int dy^3 \epsilon_{\alpha\beta\gamma} J^{(p)\beta a}(y)\frac{(x-y)^\gamma}{|x-y|^3} + \partial_\alpha f^a(x), \tag{32}$$

where $J^{(p)\beta a}(y)$ represents the r.h.s. of eq. (30) (or (31)) divided by 2, with $f^a(x)$ being an arbitrary function that takes into account the undeterminacy of the longitudinal part of $B^{(p)}{}^{a}_{\alpha}(x)$.

Once the equations of motion are solved perturbativelly, one must consider the action on-shell, which may be writen down as a power series too

$$\begin{aligned} S_{os} &= S_{os}^{CS} \\ &= \frac{\Lambda}{2}\int d^3x\, \epsilon^{\mu\nu\rho}\,(B^a_\mu \partial_\nu B^a_\rho + \frac{\Lambda}{3}\, f^{abc} B^a_\mu B^b_\nu B^c_\rho)|_{on-shell} \\ &= \frac{\Lambda}{2}\sum_{p=0}^{\infty} S^{(p)}\Lambda^p, \end{aligned} \tag{33}$$

with

$$S^{(p)} = \int d^3x\, \epsilon^{\mu\nu\rho}\left(\sum_{r,s}^{r+s=p}(B^{(r)}{}^{a}_{\mu}\partial_\nu B^{(s)}{}^{a}_{\mu}) + \frac{1}{3} f^{abc}\sum_{r,s,q}^{r+s+q=p-1}(B^{(r)}{}^{a}_{\mu} B^{(s)}{}^{b}_{\nu} B^{(q)}{}^{c}_{\rho})\right), \tag{34}$$

as can be verified after some algebra. From equations (30)-(34) we can obtain with a moderate effort the first two contributions to S_{os}. Firstly, we use eq.(32) to write the solution of eq.(31) as

$$B^{(0)}{}^{a}_{\alpha}(x) = \sum_{i=1}^{n} D_{i\alpha}(x) I^a_i(0), \tag{35}$$

where we have defined

$$D_{i\alpha}(x) \equiv \frac{1}{4\pi}\oint_{\gamma_i} dz^\gamma \frac{(x-z)^\beta}{|x-z|^3}\,\epsilon_{\alpha\beta\gamma}. \tag{36}$$

In the expression for $B^{(0)}{}^{a}_{\alpha}$ we have omited the gradient $\partial_\alpha f^a$, which does not contribute to the first two terms of S_{os}, as we shall see later. The $0-th$ order contribution to S_{os} is then given by

$$\begin{aligned} S^{(0)} &= \int d^3x \epsilon^{\mu\nu\rho} B^{(0)}{}^{a}_{\mu}\partial_\nu B^{(0)}{}^{a}_{\rho} \\ &= \frac{1}{4}\sum_{i,j} I^a_i(0) I^a_j(0) L(i,j), \end{aligned} \tag{37}$$

where

$$L(i,j) \equiv \frac{1}{4\pi}\oint_{\gamma_i} dz^\mu \oint_{\gamma_j} dy^\rho \frac{(z-y)^\beta}{|z-y|^3}\,\epsilon_{\mu\nu\rho}, \tag{38}$$

is the Gauss Linking Number (GLN) of γ_i and γ_j. It should be said that this expression is not well defined when $i = j$, although it can be converted into a meaningful expression by applying certain regularization procedure, even in this case [2]. From eq.(37) it is evident that the gradient $\partial_\alpha f^a$ does not contribute up to this order. Also, we see that the first line in eq.(37) is just the Abelian CS action, and the second one is precisely the action OS obtained for the Abelian CS action coupled to external particles that carry Abelian charges [5].

Regarding the first order contribution to the action OS, one finds, from the general results discussed above, the following expression

$$S^{(1)} = \int d^3x\, \epsilon^{\mu\nu\rho}\left(2B^{(0)a}_{\ \ \mu}\partial_\nu B^{(1)a}_{\ \ \rho} + \frac{1}{3}f^{abc}(B^{(0)a}_{\ \ \mu}B^{(0)b}_{\ \ \nu}B^{(0)c}_{\ \ \rho})\right). \tag{39}$$

Observe that $B^{(1)a}_{\ \ \rho}$ enters in this expression (see the first term) just through its rotational, which is given by eq. (30) as

$$\begin{aligned}\epsilon^{\mu\nu\rho}\,\partial_\nu B^{(1)a}_{\ \ \rho}(x) &= -\frac{1}{2}\,\epsilon^{\mu\nu\rho}\, f^{abc}B^{(0)b}_{\ \ \nu}(x)B^{(0)c}_{\ \ \rho}(x)\ -\\ &-\ \frac{1}{2}\sum_{i=1}^{n}\oint_{\gamma_i} dz^\mu \int_0^z dz_1^{\mu_1} R^{(0)aa_1}_{\ \ \ \mu_1}(z_1)\delta^3(x-z)\, I_i^{a_1}(0).\end{aligned} \tag{40}$$

Then, up to this order, we do not have to solve the equation for $B^{(1)a}_{\ \ \rho}$. Putting all together, we finally find

$$\begin{aligned}S^{(1)} = -\frac{1}{4}\sum_{i,j,k} f^{abc}I_i^a(0)I_j^b(0)I_k^c(0)\Big\{\frac{1}{3}\int d^3x\,\epsilon^{\mu\nu\rho}D_{i\mu}(x)D_{j\nu}(x)D_{k\rho}(x)\ +\\ +\oint_{\gamma_i} dz^\mu \int_0^z dy^\nu D_{j\mu}(z)D_{k\nu}(y)\Big\}.\end{aligned} \tag{41}$$

This expression vanishes when the isovectors $I_i^a(0)$, $I_j^b(0)$, and $I_k^c(0)$ are linearly dependent. To interpret our results, we shall consider the simplest (non-trivial) case: let us assume that there are just three particles of Wong, carrying independent isovectors at $\tau = 0$. Furthermore, let us also take $SU(2)$ as gauge group, and set $I_i^a(0) = \delta_i^a$. Under these assumptions, $S^{(1)}$ can be writen as

$$\begin{aligned}S^{(1)}(1,2,3) = &-\frac{1}{2}\int d^3x\,\epsilon^{\mu\nu\rho}D_{1\mu}(x)D_{2\nu}(x)D_{3\rho}(x)\ -\\ &-\frac{1}{2}\int d^3x\int d^3y\Big\{T_1^{[\mu,\nu]}(x,y)D_{2\mu}(x)D_{3\nu}(y)\,+\\ &\qquad +T_2^{[\mu,\nu]}(x,y)D_{3\mu}(z)D_{1\nu}(y)\,+\\ &\qquad +T_3^{[\mu,\nu]}(x,y)D_{1\mu}(z)D_{2\nu}(y)\Big\},\end{aligned} \tag{42}$$

where we have introduced the bilocal tensor density associated to the curve γ_i

$$T^{\mu,\nu}_{\gamma_i}(x,y) \equiv \oint_{\gamma_i} dz^\mu \int_0^z dz'^\nu \delta^3(x-z)\delta^3(y-z'). \tag{43}$$

Observe that it is the antisymmetric part (in μ ,ν) of this object which enters in the expression for $S^{(1)}(1,2,3)$. Also, $S^{(1)}(1,2,3)$ is antisymmetric under interchanges of the curves 1,2,3.

Expression (42) is, up to a factor, The Third Milnor Linking Coeficient (TMLC) [8]. We recognized it from reference [9], where this link invariant appears as a contribution to the vacuum expectation value of the product of Wilson Loops, in the context of perturbative Quantum non-Abelian CS theory. The TMLC is a highly non-trivial link invariant associated to three non-intersecting closed curves. It is defined whenever the three curves do not link each other in the Gauss sense, i.e.

$$L(i,j) \neq 0, \qquad \forall i,j, \qquad i \neq j. \tag{44}$$

In fact, the TMLC follows the Gauss Linking Number (which is then the Second MLC) in an infinite sequence of link invariants discovered by Milnor, the so called Higher Order Linking Coefficients K_n. The $p-th$ coefficient makes sense only if $K_p = 0$, for $p < n$ [8].

It is interesting to see how condition (44) arises in our scheme. First, observe that the $0-th$ order equation of motion eq.(31) is trivially integrable, since its r.h.s. has vanishing divergence. For the next order, (see eq.(40)) the corresponding integrability condition (which again is obtained by taking the divergence on both sides) is found to be

$$\sum_{i,j} f^{abc} I_i^b(0) I_j^c(0)\, \delta^3(x - z_i(0)) \int_{\gamma_i} dz^\mu D_{j\mu}(z) = 0. \tag{45}$$

Under the simplifications that lead to eq. (42) (three loops, $N = 2$, and $I_i^a(0) = \delta_i^a$), eq. (45) may be writen as

$$\Big(\delta^3(x - z_i(0) - \delta^3(x - z_j(0))\Big) L(i,j) = 0, \tag{46}$$

for any pair i,j, with $i \neq j$. If the curves do not intersect each other, this equation just tells us that $L(i,j) = 0$, as expected. Thus, we obtain that the consistence condition under which the first order action OS is meaningful, is precisely the existence condition for the TMLN.

IV. GENERAL COVARIANCE OF THE TMLN AND CONSISTENCE CONDITIONS

Expression (42) is not manifestly invariant under diffeomorphisms, since the kernel $\frac{(x-y)^\mu}{|x-y|^3}$ that enters in the definition of $D_{i\mu}(x)$ does not transforms covariantly under general coordinate changes. The same observation applies to expression (44) for the GLN. It could be said that to solve the metric-independent equations of motion one has introduced a particular metric, the Euclidean one, that breaks the general covariance of the action OS. This can be better understood by analogy with gauge theories, where it is frecuent to deal with non manifestly gauge-invariant expressions for gauge-invariant quantities, once the gauge has been fixed. In view of this it will be interesting to have generally covariant expressions for our link invariants, that allow us both to see explicitly that they are metric independent, and to dispose of an appealing interpretation of their geometrical meaning.

With this goal in mind, we find it useful to define the following sequence of differential forms. To a volume V of R^3, we can associate the $0-$ form

$$f(y,V) \equiv \int_V d^3x\,\delta^3(x-y), \tag{47}$$

with support in V. Under a general coordinate transformation, the Jacobian that rises from the volume element compensates the inverse of the Jacobian produced by the Dirac's delta function. Hence, $f(y,V)$ transforms covariantly under diffeomorphisms. As we shall see, this is a common feature of all the forms we are going to build up from $f(y,V)$. Also, it is worth observing that these forms are metric-independent. Taking the opposite of the exterior derivative of $f(y,V)$ we define the $1-$ form

$$\begin{aligned} g_\mu(y,\partial V) &\equiv -\frac{\partial}{\partial y^\mu} f(y,V) \\ &= \frac{1}{2}\int_{\partial V} d\Sigma^{\nu\rho}(x)\epsilon_{\mu\nu\rho}\,\delta^3(x-y), \end{aligned} \tag{48}$$

where ∂V is the boundary of V and we have used Stokes Theorem to produce the second line in the r.h.s. of this equation. The $1-$ form $g(y,\partial V) = -df(y,V)$ is also metric-independent and generally-covariant. Expression (48) also serves to define a $1-$ form $g_\mu(y,\Sigma)$ for arbitrary (i.e., not necessarily closed) surfaces Σ

$$g_\mu(y,\Sigma) \equiv \frac{1}{2}\int_{\Sigma} d\Sigma^{\nu\rho}(x)\epsilon_{\mu\nu\rho}\,\delta^3(x-y), \tag{49}$$

that enjoys the same transformation properties of $g_\mu(y,\partial V)$. Taking the exterior derivative of this object we obtain in turn

$$\begin{aligned} h_{\mu\nu}(y,\partial\Sigma) &\equiv 2\partial_{[\mu}g_{\nu]}(y,\Sigma) \\ &= \epsilon_{\mu\nu\rho}\oint_{\partial\Sigma} dx^\rho\delta^3(x-y), \end{aligned} \tag{50}$$

where we have employed Stokes Theorem again. The $2-$ form $h(y,\partial\Sigma) = dg(y,\Sigma)$ can also be extended to open curves γ. In that case,

$$h_{\mu\nu}(y,\gamma) \equiv \epsilon_{\mu\nu\rho}\oint_{\gamma} dx^\rho\delta^3(x-y). \tag{51}$$

From h we can define the vector density

$$T^\mu_\gamma(y) \equiv \frac{1}{2}\epsilon^{\mu\nu\rho}\;h_{\nu\rho}(y,\gamma) = \oint_\gamma dx^\mu\delta^3(x-y), \tag{52}$$

that precedes to the bilocal density $T^{\mu\nu}_\gamma(x,y)$ defined in eq. (43) in an infinite list of "loop coordinates" with well studied properties [10]. For our purposes, it suffices to notice that both objects are metric-independent densities, and that $T^{\mu\nu}$ obeys the "differential constraint"

$$\begin{aligned} \frac{\partial}{\partial x^\mu}T^{\mu\nu}_\gamma(x,y) &= \Big(-\delta^3(x-x_0)+\delta^3(x-y)\Big)T^\nu_\gamma(y) \\ \frac{\partial}{\partial y^\nu}T^{\mu\nu}_\gamma(x,y) &= \Big(\delta^3(y-x_0)-\delta^3(y-x)\Big)T^\mu_\gamma(x), \end{aligned} \tag{53}$$

and the "algebraic constraint"

$$T_{\gamma}^{(\mu\nu)}(x,y) \equiv \frac{1}{2}\Big(T_{\gamma}^{\mu\nu}(x,y) + T_{\gamma}^{\nu\mu}(x,y)\Big) = T_{\gamma}^{\mu}(x)T_{\gamma}^{\nu}(y). \tag{54}$$

To close the sequence, we take the opposite of the exterior derivative of the 2− form $h_{\mu\nu}(y,\gamma)$, which defines then a 3− form with support on the boundary $\partial\gamma$ as

$$\begin{aligned} i_{\mu\nu\rho}(y,\partial\gamma) &\equiv -3\partial_{[\mu}h_{\nu\rho]}(y,\partial\gamma) \\ &= \epsilon_{\mu\nu\rho}\Big(\delta^3(y-x_f) - \delta^3(y-x_0)\Big), \end{aligned} \tag{55}$$

with x_f and x_0 being the starting and ending points of γ. Finally, observe that, as in the previous cases, the 3− form $i(y,\partial\gamma) = -dg(y,\gamma)$ may also be trivially extended to the case of "open 0− spheres", i.e., single points

$$i_{\mu\nu\rho}(y,x) \equiv \epsilon_{\mu\nu\rho}\delta^3(y-x). \tag{56}$$

Now, let us consider the quantity

$$\begin{aligned} I(\Sigma_1,\Sigma_2,\Sigma_3) = &\int d^3x\, \epsilon^{\mu\nu\rho} g_{1\mu}(x)g_{2\nu}(x)g_{3\rho}(x) \; + \\ &+ \int d^3x \int d^3y \Big\{T_1^{[\mu,\nu]}(x,y)g_{2\mu}(x)g_{3\nu}(y) + \\ &\qquad +T_2^{[\mu,\nu]}(x,y)g_{3\mu}(x)g_{1\nu}(y) + \\ &\qquad +T_3^{[\mu,\nu]}(x,y)g_{1\mu}(x)g_{2\nu}(y)\Big\}, \end{aligned} \tag{57}$$

with Σ_1, Σ_2 and Σ_3 being arbitrary surfaces. $g_{i\mu}(x)$ is just a shorthand for $g_\mu(x,\Sigma_i)$. Also, by $T_i^{[\mu,\nu]}(x,y)$ we mean the (antisymmetrized) bilocal density defined before, evaluated at the boundary $\partial\Sigma_i$ of Σ_i. Due to the metric independence and general covariance of the ingredients of expression (57), it is inmediate to see that each one of its terms is a topological invariant of the surfaces Σ_i. The first term measures how many times the three surfaces intersect at a common point. The second term counts the oriented number of times that the boundary $\partial\Sigma_1$ crosses first the surface Σ_2 and then Σ_3. The remaining terms have a similar interpretation. Clearly, every one of these quantities is invariant under continuous deformations of R^3. What it is by no means trivial is to see wether or not they are link rather than surfaces invariants. To study this point, let us compute how $I(\Sigma_1,\Sigma_2,\Sigma_3)$ changes when Σ_1 is replaced by another surface Σ_1', such that both surfaces share the same boundary: $\partial\Sigma_1$ = $\partial\Sigma_1'$. One has

$$\begin{aligned} \Delta I_1 &\equiv I(\Sigma_1',\Sigma_2,\Sigma_3) - I(\Sigma_1,\Sigma_2,\Sigma_3) \\ &= \int d^3x\, \epsilon^{\mu\nu\rho}\Delta g_{1\mu}(x)g_{2\nu}(x)g_{3\rho}(x) \; + \\ &\quad + \int d^3x \int d^3y \Big\{T_2^{[\mu,\nu]}(x,y)g_{3\mu}(x)\Delta g_{1\nu}(y) + \\ &\qquad +T_3^{[\mu,\nu]}(x,y)\Delta g_{1\mu}(x)g_{2\nu}(y)\Big\}, \end{aligned} \tag{58}$$

where

$$\Delta g_{1\mu} \equiv g_{1\mu}(\Sigma_1') - g_{1\mu}(\Sigma_1) = g_{1\mu}(\partial V), \tag{59}$$

V being the volume enclosed by the surface that results of the composition of Σ_1' with the opossite of Σ_1: $\partial V \equiv \Sigma_1' - \Sigma_1$. Now, eq.(48) tells us that $g(\partial V) = -df(V)$. Then we can write, after integrating by parts, the second term of the second equality in eq. (58) as

$$\int d^3x \int d^3y \frac{\partial}{\partial y^\nu}\Big(T_2^{[\mu,\nu]}(x,y)\Big) g_{3\mu}(x) f_1(y) = f_1(x_0^{(2)}) \int d^3x T_2^\mu(x) g_{3\mu}(x) - \\ - \int d^3x f_1(x) T_2^\mu(x) g_{3\mu}(x). \tag{60}$$

In writing this equation, we have also employed the differential constraints eq.(53) obeyed by $T^{[\mu,\nu]}(x,y)$. In a similar form, the last term of eq. (58) may writen as

$$\int d^3x \int d^3y \frac{\partial}{\partial x^\mu}\Big(T_3^{[\mu,\nu]}(x,y)\Big) g_{2\nu}(y) f_1(x) = -f_1(x_0^{(3)}) \int d^3x T_3^\mu(x) g_{2\mu}(x) + \\ + \int d^3x f_1(x) T_3^\mu(x) g_{2\mu}(x). \tag{61}$$

Expressions (60) and (61) are related, since in view of the definitions of $h_{\mu\nu}$ and T^μ one has

$$\int d^3x f_1(x) T_3^\mu(x) g_{2\mu}(x) = \int d^3x f_1(x) \epsilon^{\mu\nu\rho} \partial_\nu g_{3\rho}(x) g_{2\mu}(x) \\ = \int d^3x \Big[-\epsilon^{\mu\nu\rho} \Delta g_{1\mu}(x) g_{2\nu}(x) g_{3\rho}(x) + \int d^3x f_1(x) T_2^\mu(x) g_{3\mu}(x) \Big]. \tag{62}$$

Then, substituting eqs. (60) and (61) into (58), and using the identity (62) we finally get

$$\Delta_1 I(\Sigma_1, \Sigma_2, \Sigma_3) = \Big(f_1(x_0^{(2)}) - f_1(x_0^{(3)})\Big) L(\partial\Sigma_2, \partial\Sigma_3), \tag{63}$$

where we have also employed that the GLN $L(\partial\Sigma_2, \partial\Sigma_3)$ is equal to the "crossing number" of $\partial\Sigma_2$ with Σ_3, and that it is symmetric under exchange of Σ_2 with Σ_3 [11]

$$L(\partial\Sigma_2, \partial\Sigma_3) = \int d^3x T_2^\mu(x) g_{3\mu}(x) = \int d^3x T_3^\mu(x) g_{2\mu}(x). \tag{64}$$

Hence, from eq. (63) we find that $\Delta_1 I$ vanishes provided that the curves 2 and 3 do not intersect and have vanishing GLN. These arguments can be obviously repeated to calculate the dependence of $I(\Sigma_1, \Sigma_2, \Sigma_3)$ on the surfaces Σ_2 and Σ_3, with similar conclusions. The result is then that $I(\Sigma_1, \Sigma_2, \Sigma_3)$ is a link invariant of the curves that bound the surfaces Σ_i (besides being a "surfaces-invariant" quantity) provided that these curves does not cross each other and have vanishing GLN.

This result, together with the appearance of $I(\Sigma_1, \Sigma_2, \Sigma_3)$ leads us naturally to ask which is the relation, if any, between this quantity and the first order contribution to S_{os} (eq.(42)) obtained in the previous section. To seek for the precise relation between these quantities, let us observe that the key for stablishing the independence of $I(\Sigma_1, \Sigma_2, \Sigma_3)$ of the surfaces, was the fact that it does not vary when the $g_\mu(\Sigma)$'s change by additive gradients $\partial_\mu \Lambda$ (provided the GLN's of the boundaries $\partial\Sigma$'s vanish). But a direct calculation shows that

$$\partial_\mu D_{i\nu}(x) - \partial_\nu D_{i\mu}(x) = \partial_\mu g_{i\nu}(x) - \partial_\nu g_{i\mu}(x), \tag{65}$$

thus, $D_{i\nu}$ and g_μ just differ by a gradient and one may replace the latter by the former in expression (57). Hence, one has

$$S^1(\partial\Sigma_1, \partial\Sigma_2, \partial\Sigma_3) = -\frac{1}{2} I(\Sigma_1, \Sigma_2, \Sigma_3) + \Delta, \tag{66}$$

where Δ is a function of the surfaces that vanishes when the GLN are equal to zero. This equation provides the relation we were looking for. In passing, we see that by the same argument it is allowed to neglect the gradient $\partial_\alpha f^a$ that comes from the "Biot-Savart" solution also to compute the first order contribution to S_{os}.

The result we have obtained in this section could be summarized as follows. It is posible to provide two equivalent analytical expressions for the TMLC of a set of three curves. One of them is not manifestly invariant under diffeomorphisms and is given explicitly by $-2S^1(1,2,3)$. In turn, the other one, given by $I(\Sigma_1, \Sigma_2, \Sigma_3)$ is manifestly invariant under diffeomorphisms, but it is not explicitly "link-dependent"; instead, it is "surface-dependent". Both expression are related through a geometric mechanism: changing from the "link" to the "surfaces" presentation, amounts to performing an "Abelian gauge transformation" under which the TMLN is invariant.

V. DISCUSSION

We have presented a method for obtaining link invariants through the study of the classical equations of motion of non-Abelian Chern-Simons theory coupled to linked sources. The method relies on the fact that the classical action that we take should retain its topological character when it is calculated on-shell. Furthermore, a simple argument allows to see that this is true even perturbativelly. We have studied the first two invariants that the method provides. While the first one is rather trivial (in the sense that it appears in almost all the discussions about link invariants and Chern-Simons theory), the second one is highly non-trivial, and corresponds to the Third Milnor Linking Coeficient [8]. This invariant is useful, for instance, to characterize the entanglement properties of the Borromean Rings [8,11], which constitute a non-trivial three-component link that has vanishing Gauss Linking Number between any pair of its components.

We have also introduced a geometrical setting that allows us to write down the TMLN in a manifestly diffeomorphism invariant form, althougth in this presentation this object looks like a "surfaces" rather than a "link" invariant. The surfaces appearing in this presentation are such that their boundaries are the components of the link (they are Seifert Surfaces, in knot-theoretical parlance). This fact, instead of being an inconvenience, is wellcomed: it allows to interpret the TMLN as a particular combination of intersection numbers between the surfaces and their boundaries. It is interesting to point out that there is a recent work [12] devoted to the interpretation of the TMLN, with which our results should be compared.

The explicit choice of a particular set of Seifert Surfaces in the expression for the TMLN is seen to be related with a kind of Abelian Gauge Symmetry: the surfaces enter in that expression through certain 1−forms. Changing a Seifert Surface by another one, amounts to shifting its associated 1−form by a gradient, and this transformation is seen to leave the "surface-dependent" expression unchanged; thus, one obtains that the dependence in the surfaces is accomphished only through their boundaries, as corresponds to a link-invariant.

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